Grupa A, Pismeni ispit iz Matematike II, 20.06.2013.
ispit pisati isključivo hemiskom olovkom

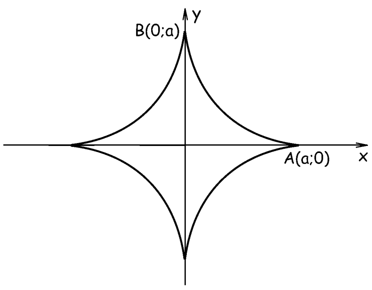
1. Izračunati površinu omotača tijela koje nastaje kada dio krive $y = x^3$, koji se nalazi između pravih $x = -\frac{2}{3}$ i $x = \frac{2}{3}$, rotira oko x-ose.

2. Uvođenjem sfernih koordinata izračunati integral $\int\limits_0^1 \mathrm{d}x \int\limits_0^{\sqrt{1-x^2}} \mathrm{d}y \int\limits_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2 \mathrm{d}z.$

3. Izračunati vrijednost krivoliniskog integrala $I = \oint\limits_C y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z$ duž zatovorene krive C koja je dobijena kao presjek sljedećih površina: $x^2 + y^2 = r^2$ i $x^2 = rz$ $(r > 0)$. (Kriva C je orjentisana pozitivno ako se posmatra sa z-ose za $z > r$).

4. Izračunati površinski integral $I = \iint\limits_{S+} x^2\mathrm{d}y\mathrm{d}z + y^2\mathrm{d}z\mathrm{d}x + z^2\mathrm{d}x\mathrm{d}y$ gdje je $S+$ spoljašnja strana kupe određena omotačem $z^2 = x^2 + y^2$, $0 \le z \le h$ i osnovom $x^2 + y^2 \le h^2$, $z = h$ za fiksirano $h > 0$.

Grupa B, Pismeni ispit iz Matematike II, 20.06.2013.
ispit pisati isključivo hemiskom olovkom



1. Izračunati površinu omotača tijela koje nastaje kada astroida $x = a\cos^3 t$, $y = a\sin^3 t$ rotira oko x-ose (grafik astroide je prikazan na slici desno).

2. Izračunati integral $\iiint\limits_V xyz\mathrm{d}x\mathrm{d}y\mathrm{d}z$ gdje je oblast V ograničena sferom $x^2 + y^2 + z^2 = 1$ i ravnima $x = 0$, $y = 0$, $z = 0$ u I oktantu.

3. Uz pomoć formule Stoksa, izračunati krivolinijski integral $I = \oint\limits_L x^2y^3\mathrm{d}x + \mathrm{d}y + z\mathrm{d}z$ gdje je L krug dat sa $x^2 + y^2 = r^2$ i $z = 0$ (r>0). (L je pozitivno orjentisana kriva ukoliko se posmatra sa pozitivnog dijela z-ose.)

4. Izračunati površinski integral $I = \iint\limits_{S+} y^2\mathrm{d}y\mathrm{d}z + (y^2 + x^2)\mathrm{d}z\mathrm{d}x + (y^2 + x^2 + z^2)\mathrm{d}x\mathrm{d}y$ gdje je $S+$ spoljašnja strana polusfere $x^2 + y^2 + z^2 = 2Rx$, $z > 0$ (za fiksirano $R > 0$).

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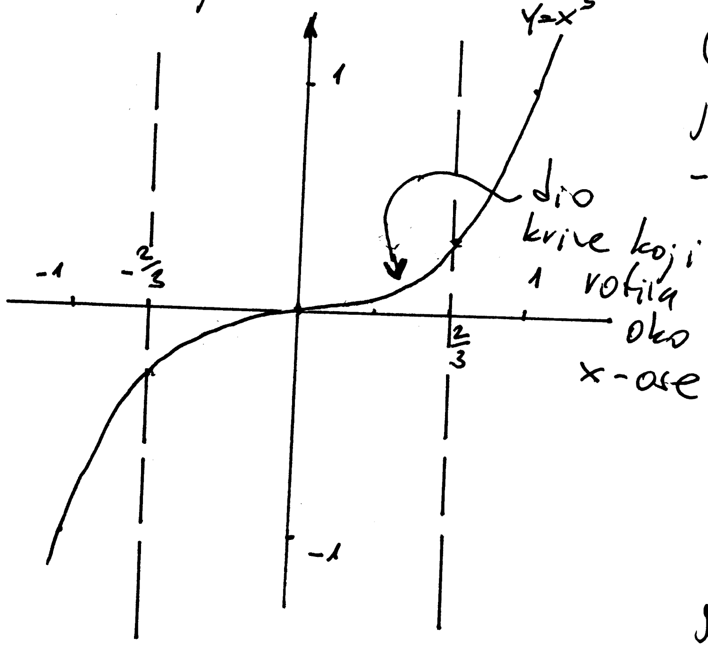
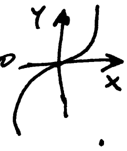
\# Izračunati površinu omotača tijela koje nastaje kada dio krive $y=x^3$, koji se nalazi između pravih $x=-\frac{2}{3}$ i $x=\frac{2}{3}$, rotira oko x-ose.

Rj. Skicirajmo sliku



Znamo da kriva $y=x^3$ izgleda ovako



U našem slučaju dovoljno ju je nacrtati u granicama od -1 do 1.

Prisjetimo se formule

$$P=2\pi\int_{t_1}^{t_2}|\mu(t)|\sqrt{[\eta'(t)]^2+[\mu'(t)]^2}\,dt$$

gdje je $c:\begin{cases} x=\eta(t)\\ y=\mu(t)\\ t_1\le t\le t_2\end{cases}$

Naša kriva je

$$c:\begin{cases} y=x^3\\ -\frac{2}{3}\le x\le \frac{2}{3}\end{cases}$$

ili napisano u drugačijem obliku $c:\begin{cases} x=t\\ y=t^3\\ -\frac{2}{3}\le t\le\frac{2}{3}\end{cases}$

Kako je $y'=3x^2$ imamo

$$P=2\pi\int_{-2/3}^{2/3}|x^3|\sqrt{1+3x^2}\,dx=2\pi\cdot 2\int_0^{2/3}\underbrace{x^3}_{x^2\cdot x}\sqrt{1+(3x^2)^2}\,dx=4\pi\int_0^{2/3}x^3\sqrt{1+9x^4}\,dx$$

$$=\left|\begin{array}{l} 1+9x^4=z\\ 36x^3dx=dz \quad x\Big|_0^{\frac{2}{3}}\Rightarrow z\Big|_1^{\frac{25}{9}}\\ x^3dx=\frac{1}{36}dz\end{array}\right|=4\pi\cdot\frac{1}{36}\int_1^{25/9}t^{\frac{1}{2}}dt=\frac{\pi}{9}\cdot\frac{2}{3}t^{\frac{3}{2}}\Big|_1^{\frac{25}{9}}=$$

$$=\frac{2\pi}{27}\left(\frac{125}{27}-1\right)$$ traženo rješenje

Uvođenjem sfernih koordinata izračunati integral

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2\, dz$$

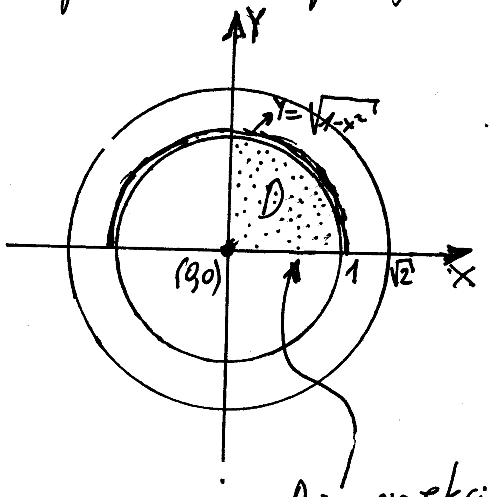
Rj. Ako sa Ω označimo oblast integracije vidimo da imamo

$$\Omega : \begin{cases} 0 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \\ \sqrt{x^2+y^2} \le z \le \sqrt{2-x^2-y^2} \end{cases}$$

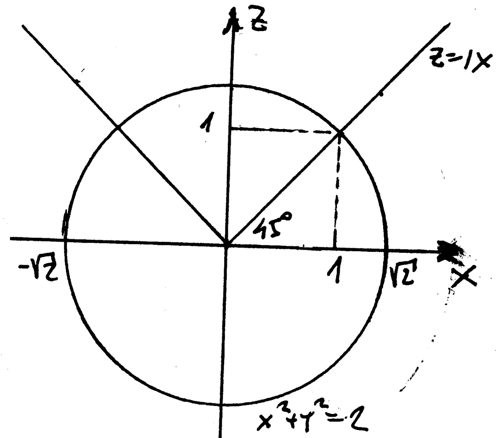
Iz $\sqrt{x^2+y^2} \le z \le \sqrt{2-x^2-y^2}$ čitamo da su date dvije površi

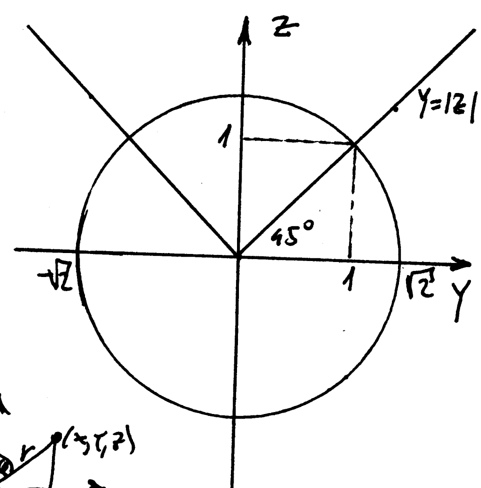
$x^2+y^2=z^2$ (kupa) i $x^2+y^2+z^2=2$ (sfera)

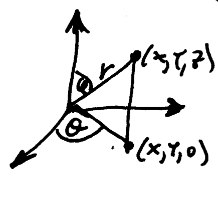
Napravimo presjek oblasti Ω sa xOy, xOz i yOz ravni.



D je projekcija od Ω na xOy ravan







opis tačke

Ako uvedemo sferne koordinate

$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

$dx\,dy\,dz = r^2\sin\varphi\, dr\, d\varphi\, d\theta$

$$\Omega \xrightarrow{\text{transformiše}} \Omega' : \begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le \varphi \le \frac{\pi}{4} \\ ??? \le r \le ??? \end{cases}$$

vidimo sa projekcije

Sa slika presjeka datih površina sa koordinatnim ravninama vidimo da će r uzimati sve tačke koje se nalaze između čunja i sfere.
Pa da bi odredili granice za r posmatrajmo formulu

$$z \le \sqrt{2-x^2-y^2}$$

$$z^2 \le 2-x^2-y^2$$

$$x^2+y^2+z^2 \le 2 \quad \Rightarrow \quad r^2 \le 2$$

$$r \le \sqrt{2}$$

što se slaže sa nacrtanim slikama

$$\Omega' : \begin{cases} 0 \le r \le \sqrt{2} \\ 0 \le \varphi \le \frac{\pi}{4} \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}$$

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2\, dz = \iiint_\Omega z^2\, dx\,dy\,dz \;\left|\; \begin{matrix}\text{uvodimo} \\ \text{sferne} \\ \text{koordinate}\end{matrix} \right| = \iiint_{\Omega'} r^2\cos^2\varphi\, r^2 \sin\varphi \; dr\,d\varphi\,d\theta$$

$$= \int_0^{\sqrt{2}} r^4\, dr \int_0^{\pi/4} \underbrace{\cos^2\varphi \sin\varphi\, d\varphi}_{\cos^2\varphi\,(-1)\,d(\cos\varphi)} \int_0^{\pi/2} d\theta = \theta\Big|_0^{\pi/2} \cdot \frac{-1}{3}\cos^3\varphi\Big|_0^{\frac{\pi}{4}} \cdot \frac{1}{5} r^5\Big|_0^{\sqrt{2}} =$$

$$= -\frac{1}{15}\cdot\frac{\pi}{2}\cdot 4\sqrt{2}\left(\left(\frac{\sqrt{2}}{2}\right)^3 - 1\right) = \frac{-\pi}{15}\, 2\sqrt{2}\left(\frac{2\sqrt{2}}{8} - 1\right)$$

$$= \frac{\pi}{15}\left(2\sqrt{2} - 1\right)$$

traženo rješenje

(#) Izračunati vrijednost krivoliniskog integrala

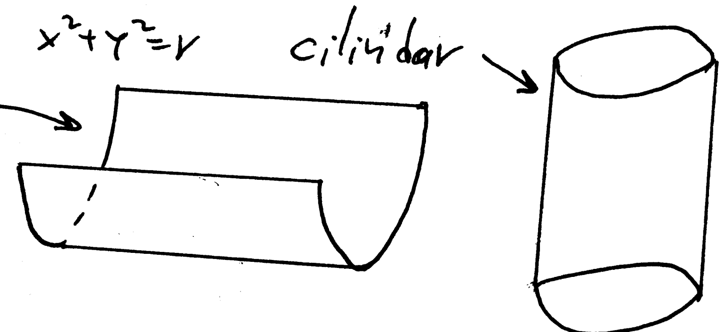
$$\oint_C y\,dx + z\,dy + x\,dz$$

duž zatvorene krive c koja je dobijena kao presjek sljedećih površina

$$x^2+y^2=r^2 \quad i \quad x^2=rz \quad (r>0)$$

(kriva c je orjentisana pozitivno ako se posmatra sa z-ose za $z>r$).

Rj. Izgled krive c nam u ovom zadatku neće pomoći da lakše uradimo zadatak, pa je nećemo skicirati. Samo primjetimo da je $x^2+y^2=r$ cilindar a da je $x^2=rz$ paraboloid i da njihov presjek proizvodi krivu c.



Prisjetimo se: Ako je $c: \begin{cases} x=\eta(t) \\ y=\mu(t) \\ z=\theta(t) \\ t_1 \le t \le t_2 \end{cases}$ tada

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \int_{t_1}^{t_2} \Big[P(\eta(t),\mu(t),\theta(t))\,\eta'(t) + Q(\eta(t),\mu(t),\theta(t))\,\mu'(t) + R(\eta(t),\mu(t),\theta(t))\,\theta'(t) \Big] dt$$

Da bismo parametrizirali datu krivu, prvo parametrizujmo krug $x^2+y^2=r^2$:

$$x^2+y^2=r^2: \begin{cases} x=r\cos\varphi \\ y=r\sin\varphi \\ 0\le\varphi\le 2\pi \end{cases}$$

a smjene za x i y uvrstimo u paraboloid:

$x^2 = rz$

$r^2\cos^2\varphi = rz$

$z = r\cos^2\varphi$

Prema tome

$$C: \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = r\cos^2\varphi \\ 0 \le \varphi \le 2\pi \end{cases}$$

Kako je $dx = -r\sin\varphi\, d\varphi$, $dy = r\cos\varphi\, d\varphi$, $dz = r \cdot 2\cos\varphi(-\sin\varphi)\, d\varphi$ to je

$$\oint_C y\,dx + z\,dy + x\,dz = \int_0^{2\pi} \left(r\sin\varphi \cdot (-r)\sin\varphi + r\cos^2\varphi \cdot r\cos\varphi + r\cos\varphi \cdot (-2)r\sin\varphi\cos\varphi\right) d\varphi$$

$$= -\int_0^{2\pi} r^2\sin^2\varphi\, d\varphi + r^2\int_0^{2\pi}\cos^3\varphi\, d\varphi - 2r^2\int_0^{2\pi}\cos^2\varphi\sin\varphi\, d\varphi = I_1 + I_2 + I_3$$

$$I_1 = -\int_0^{2\pi} r^2\sin^2\varphi\, d\varphi = \left| \begin{array}{c} 1 = \sin^2\varphi + \cos^2\varphi \\ \cos 2\varphi = \cos^2\varphi - \sin^2\varphi \\ \hline 2\sin^2\varphi = 1 - \cos 2\varphi \\ \sin^2\varphi = \frac{1}{2}(1 - \cos 2\varphi) \end{array} \right| = -\frac{r^2}{2}\int_0^{2\pi}(1 - \cos 2\varphi)\, d\varphi = \underset{\text{za vježbu}}{\ldots} = -r^2\pi$$

$$I_2 = r^2\int_0^{2\pi}\cos^3\varphi\, d\varphi = r^2\int_0^{2\pi}\cos\varphi(1 - \sin^2\varphi)\, d\varphi = r^2\int_0^{2\pi}(1 - \sin^2\varphi)\, d(\sin\varphi) = \underset{\text{za vježbu}}{\ldots} = 0$$

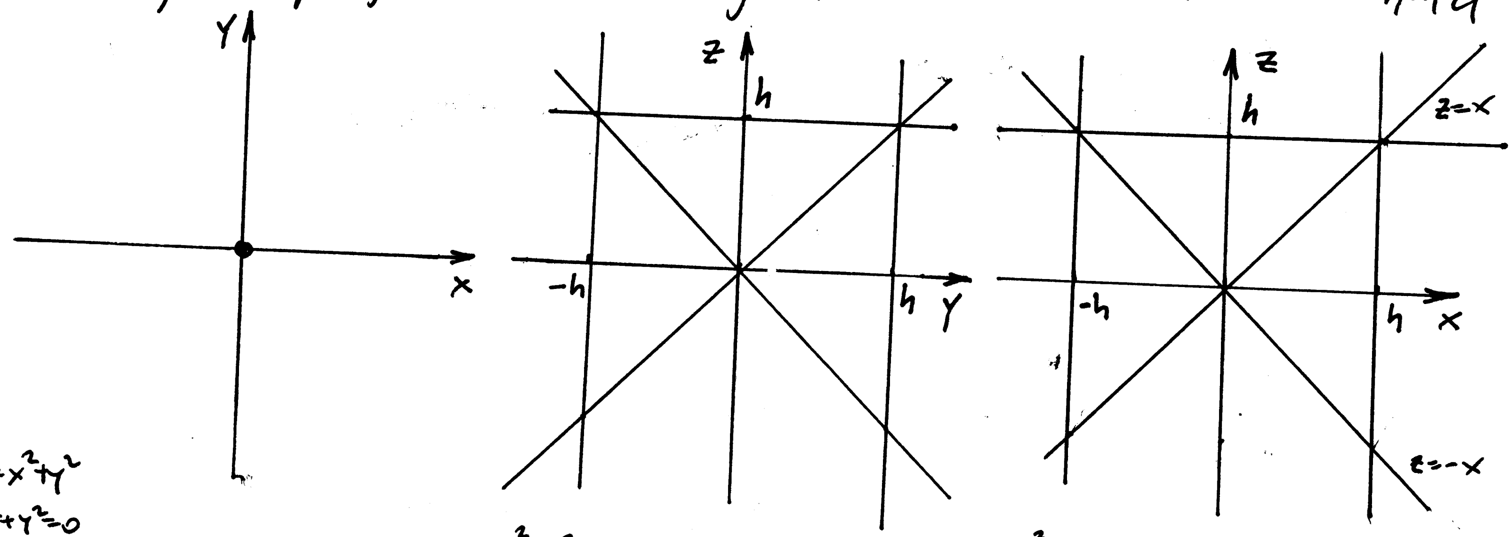
$$I_3 = -2r^2\int_0^{2\pi}\cos^2\varphi\sin\varphi\, d\varphi = \left| d(\cos\varphi) = -\sin\varphi\, d\varphi \right| = 2r^2\int_0^{2\pi}\cos^2\varphi\, d(\cos\varphi) = \underset{\text{za vježbu}}{\ldots} = 0$$

Prema tome $\oint_C y\,dx + z\,dy + x\,dz = -r^2\pi$ traženo rješenje

(#) Izračunati površinski integral $\iint\limits_{S^+} x^2\,dy\,dz + y^2\,dz\,dx + z^2\,dx\,dy$

gdje je S^+ spoljašnja strana kupe određene omotačem $z^2=x^2+y^2$, $0\le z\le h$ i osnovom $x^2+y^2\le h^2$, $z=h$ za fiksirano $h>0$.

Rj. Skicirajmo presjeke datih figura sa koordinatnim ravnima



$z^2=x^2+y^2$
$x^2+y^2=0$

$x^2+y^2\le h^2$
$z=h$

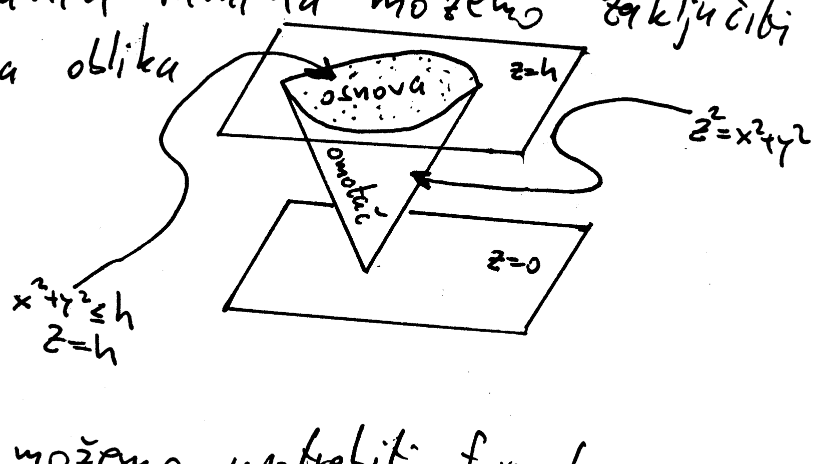
$z^2=x^2+y^2$
$y^2=z^2$
$z=\pm y$

$x^2+y^2\le h^2$
$z=h$
$y^2\le h$
$z=h$

$z^2=x^2+y^2$
$z^2=x^2$
$z=\pm x$

$x^2+y^2\le h^2$
$z=h$
$x^2\le h^2$
$z=h$

Sa presjeka figura sa koordinatnim ravnima možemo zaključiti da je $z^2=x^2+y^2$, $0\le z\le h$ kupa oblika dok je osnova u stvari "poklopac" kupe (vidi sliku)



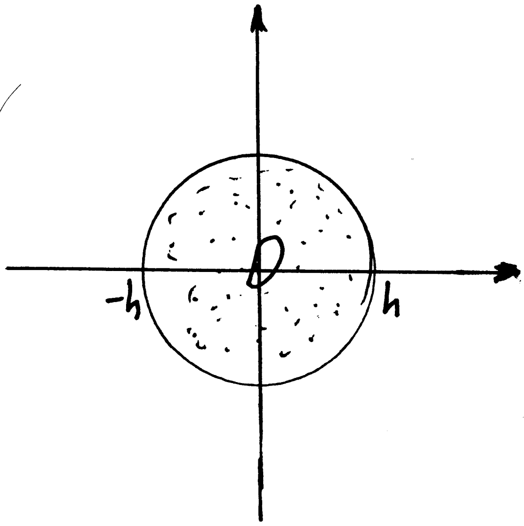
Kako imamo zatvorenu figuru možemo upotrebiti formulu Gauss-Ostrogradskog.

$$\iint\limits_{S+} x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy \overset{\text{form. Gauss-Ostr.}}{=\!=\!=}$$

$$= \iiint\limits_{\Omega} (2x+2y+2z)\,dx\,dy\,dz = 2\iiint\limits_{\Omega} (x+y+z)\,dx\,dy\,dz$$

gdje je Ω unutrašnjost kupe koja je ograničena sa ravnima $z=0$ i $z=h$ (vidi sliku).

Ortogonalna projekcija oblasti Ω na xOy-ravan je krug sa centrom u koordinatnom početku poluprečnika h.



$$I = 2\iiint\limits_{\Omega} (x+y+z)\,dx\,dy\,dz = 2\iint\limits_{D} dx\,dy \int_{\sqrt{x^2+y^2}}^{h} (x+y+z)\,dz$$

$$= 2\iint\limits_{D}\left(xz\Big|_{\sqrt{x^2+y^2}}^{h} + yz\Big|_{\sqrt{x^2+y^2}}^{h} + \frac{1}{2}z^2\Big|_{\sqrt{x^2+y^2}}^{h}\right) dx\,dy =$$

$$= 2\iint\limits_{D}\left(hx + hy + \frac{1}{2}h^2 - x\sqrt{x^2+y^2} - y\sqrt{x^2+y^2} - \frac{1}{2}(x^2+y^2)\right) dx\,dy = \left| \begin{array}{l} \text{uvedimo polarne koordinate} \\ x=\rho\cos\varphi \quad D \xrightarrow{\text{transf.}} D' \\ y=\rho\sin\varphi \\ dx\,dy=\rho\,d\rho\,d\varphi \\ x^2+y^2=\ldots=\rho^2 \end{array} \right|$$

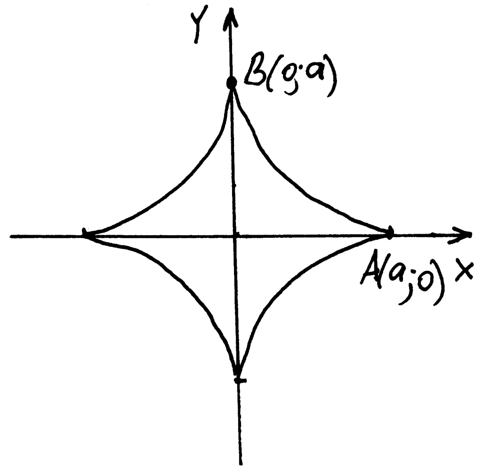
$$= 2\iint\limits_{D'}\left(h\rho\cos\varphi + h\rho\sin\varphi + \frac{1}{2}h^2 - \rho\cos\varphi\cdot\rho - \rho\sin\varphi\cdot\rho - \frac{1}{2}\rho^2\right)\rho\,d\rho\,d\varphi$$

$$= 2\int_0^{2\pi} d\varphi \int_0^{h}\left(h\rho^2\cos\varphi + h\rho^2\sin\varphi + \frac{1}{2}h^2\rho - \rho^3\cos\varphi - \rho^3\sin\varphi - \frac{1}{2}\rho^3\right) d\rho$$

$$= \underset{\ldots}{\text{za vježbu}} = \frac{h^4}{12}\int_0^{2\pi}(2\cos\varphi + 2\sin\varphi + 3)\,d\varphi = \ldots = \frac{1}{2}h^4\pi$$ traženo rješenje

Izračunati površinu omotača tijela koje nastaje kada astroida $x = a\cos^3 t$, $y = a\sin^3 t$ rotira oko x-ose.

Uputa: Grafik astroide izgleda ovako



Rj.

Tijelo koje će nastati prilikom rotacije astroide će biti simetrično u odnosu na x i y-osu.

Prema tome dovoljno je rotirati samo jedan luk astroide oko x-ose i dobijeni rezultat pomnožiti sa 2.

Prisjetimo se formule

$$P = 2\pi \int_{t_1}^{t_2} |\mu(t)| \sqrt{[\eta'(t)]^2 + [\mu'(t)]^2}\, dt \quad \text{gdje je:} \quad C: \begin{cases} x = \eta(t) \\ y = \mu(t) \\ t_1 \le t \le t_2 \end{cases}$$

U našem slučaju $\quad C: \begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \\ 0 \le t \le \frac{\pi}{2} \end{cases} \qquad \begin{aligned} x' &= 3a\cos^2 t(-\sin t) \\ y' &= 3a\sin^2 t \cos t \end{aligned}$

$$x'^2 + y'^2 = 9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t = 9a^2\cos^2 t\sin^2 t \underbrace{(\cos^2 t + \sin^2 t)}_{=1}$$

$$\sqrt{x'^2 + y'^2} = 3a\cos t\sin t$$

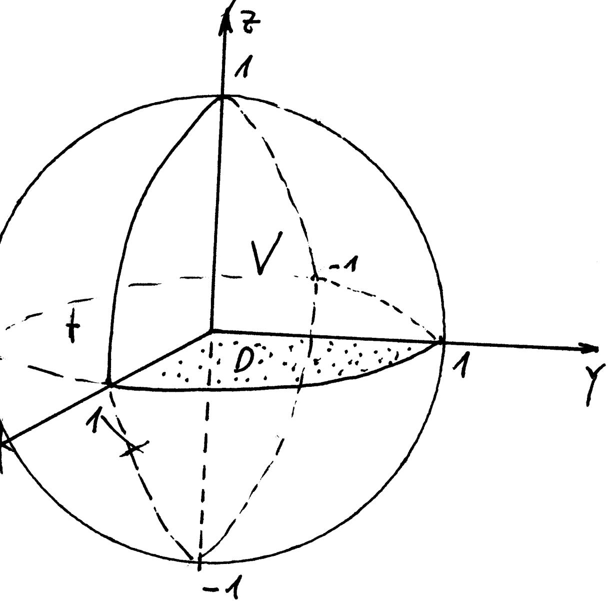
$$\frac{1}{2}P = 2\pi \int_0^{\pi/2} a\sin^3 t \cdot 3a\cos t\sin t\, dt = 6a^2\pi \int_0^{\pi/2} \sin^4 t\cos t\, dt =$$

$$= 6a^2\pi \int_0^{\pi/2} \sin^4 t\, d(\sin t) = 6a^2\pi \, \frac{1}{5}\sin^5 t \Big|_0^{\pi/2} = \frac{6a^2\pi}{5}$$

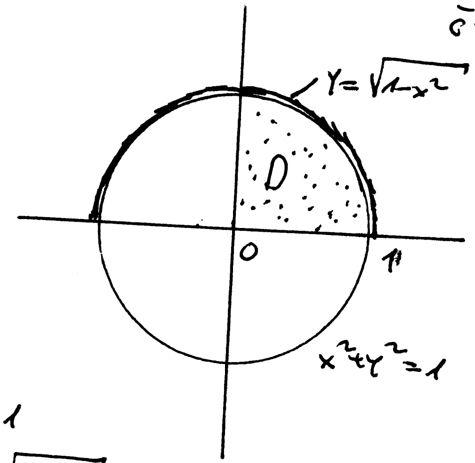
Prema tome $P = \dfrac{12a^2\pi}{5}$ tražena površina

(#) Izračunati integral $\iiint\limits_V xyz\,dx\,dy\,dz$ gdje je oblast V ograničena sferom: $x^2+y^2+z^2=1$ i ravnima $x=0$, $y=0$, $z=0$ u I oktantu.

Rj: Skiciramo oblast V



$x^2+y^2+z^2=1$ predstavlja sferu sa centrom u tački $O(0,0,0)$ poluprečnika 1. Ortogonalna projekcija date sfere na xOy ravan u I oktantu je četvrtina kruga $x^2+y^2=1$.



$y^2=1-x^2$

$y=\pm\sqrt{1-x^2}$

Prema tome $V: \begin{cases} 0\le x\le 1 \\ 0\le y\le \sqrt{1-x^2} \\ 0\le z\le \sqrt{1-x^2-y^2} \end{cases}$.

Iz dobijenih granica vidimo, da bi izračunali integral, potrebno je preći na sferne koordinate

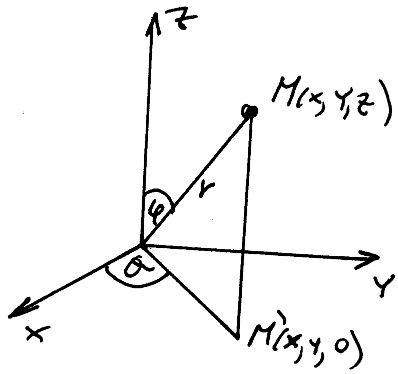
$x=r\sin\varphi\cos\theta$

$y=r\sin\varphi\sin\theta$

$z=r\cos\varphi$

$dx\,dy\,dz=r^2\sin\varphi\,dr\,d\theta\,d\varphi$

opis tačke



Uvođenjem sfernih koordinata

$V \xrightarrow{\text{transformacije}} V': \begin{cases} 0\le r\le 1 \\ 0\le\varphi\le\frac{\pi}{2} \\ 0\le\theta\le\frac{\pi}{2} \end{cases}$.

Sad imamo

$$I=\iiint\limits_V xyz\,dx\,dy\,dz = \left|\begin{array}{l}\text{uvodimo}\\ \text{sferne}\\ \text{koordinate}\end{array}\right| =$$

$$=\iiint\limits_{V'} \underline{r\sin\varphi\cos\theta}\cdot \underline{r\sin\varphi\sin\theta}\cdot \underline{r\cos\varphi}\;\underline{r^2\sin\varphi}\,dr\,d\varphi\,d\theta$$

$$=\int_0^1 r^5\,dr\int_0^{\pi/2}\sin^3\varphi\cos\varphi\,d\varphi\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta$$

$$\int_0^1 r^5\,dr = \frac{1}{6}r^6\Big|_0^1 = \frac{1}{6}$$

$$\int_0^{\pi/2}\sin^3\varphi\cos\varphi\,d\varphi = \left|\,d(\sin\varphi)=\cos\varphi\,d\varphi\,\right| = \int_0^{\pi/2}\sin^3\varphi\,d(\sin\varphi) =$$

$$=\frac{1}{4}\sin^4\varphi\Big|_0^{\frac{\pi}{2}} = \frac{1}{4}$$

$$\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \left|\,d(\sin\theta)=\cos\theta\,d\theta\,\right| = \int_0^{\pi/2}\sin\theta\,d(\sin\theta) =$$

$$=\frac{1}{2}\sin^2\theta\Big|_0^{\pi/2} = \frac{1}{2}$$

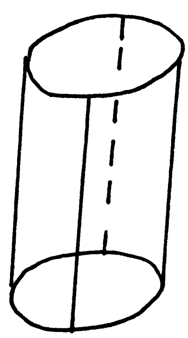
Prema tome $I=\frac{1}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{48}$ traženo rješenje

Uz pomoć formule Stoksa, izračunati krivolinijski integral

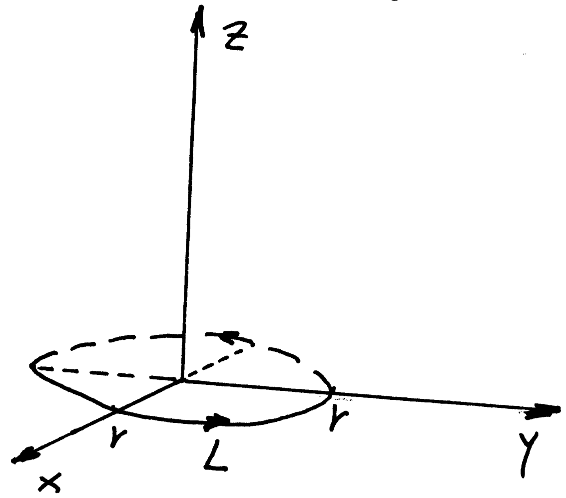
$$I=\oint_L x^2y^3\,dx+dy+z\,dz$$

gdje je L krug dat sa $x^2+y^2=r^2$ i $z=0$ $(r>0)$ (L je pozitivno orjentisana kriva ukoliko se posmatra sa pozitivnog dijela z-ose).

Rj: U xOy-ravni linija $x^2+y^2=r^2$ je krug sa centrom u tački $(0,0)$ poluprečnika r, a u prostoru to je cilindar



Krivu L u prostoru nije teško skicirati



Prisjetimo se formule Stoksa

$$\oint_C P\,dx+Q\,dy+R\,dz=\iint_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R \end{vmatrix}$$

površinski integral druge vrste

Imamo

$$I=\oint_L x^2y^3\,dx+dy+z\,dz=\iint_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2y^3 & 1 & z \end{vmatrix} =$$

$0-0$
$0-0$
$0-3x^2y^2$

$$=\iint_S (-3)x^2y^2\,dx\,dy=-3\iint_D x^2y^2\,dx\,dy=$$

| uvedimo polarne koordinate
$x=\rho\cos\varphi$
$y=\rho\sin\varphi$
$dx\,dy=\rho\,d\rho\,d\varphi$

$$D \xrightarrow{\text{transf.}} D': \begin{cases} 0\le\rho\le r\\ 0\le\varphi\le 2\pi \end{cases}$$

$$= -3 \iint\limits_{D'} \rho^2 \cos^2\varphi \, \rho^2 \sin^2\varphi \, \rho \, d\rho \, d\varphi = -3 \int_0^r \rho^5 d\rho \int_0^{2\pi} \frac{1}{4} \cdot \underbrace{4\cos^2\varphi \sin^2\varphi}_{(2\sin\varphi\cos\varphi)^2} d\varphi =$$

$$= -\frac{3}{4} \int_0^r \rho^5 d\rho \int_0^{2\pi} \sin^2 2\varphi \, d\varphi$$

$$\int_0^r \rho^5 d\rho = \frac{1}{6} \rho^6 \Big|_0^r = \frac{1}{6} r^6$$

$$\int_0^{2\pi} \sin^2 2\varphi \, d\varphi = \left| \begin{array}{l} 1 = \cos^2 2\varphi + \sin^2 2\varphi \\ \cos 4\varphi = \cos^2 2\varphi - \sin^2 2\varphi \\ \hline 2\sin^2 2\varphi = 1 - \cos 4\varphi \end{array} \right| = \frac{1}{2} \int_0^{2\pi} (1 - \cos 4\varphi) \, d\varphi = \quad \dots \quad = \pi$$

Prema tome

$$I = -\frac{\cancel{3}^{1}}{4} \cdot \frac{1}{\cancel{6}_{2}} r^6 \cdot \pi = -\frac{1}{8} r^6 \pi \quad \text{traženo rješenje}$$

Izračunati površinski integral

$$I = \iint\limits_{S+} y^2\,dy\,dz + (y^2+x^2)\,dz\,dx + (y^2+x^2+z^2)\,dx\,dy,$$

gdje je $S+$ spoljašnja strana polusfere $x^2+y^2+z^2=2Rx$, $z>0$ (za fiksirano $R>0$).

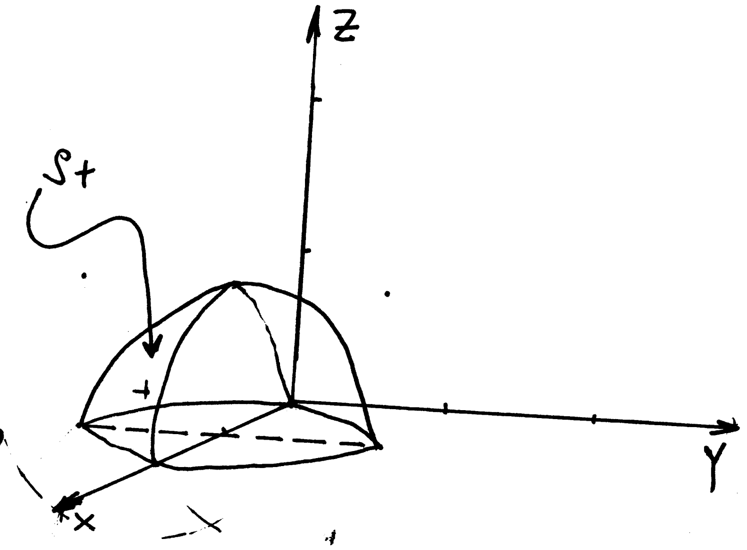
Rj.

$$x^2+y^2+z^2=2Rx$$

$$x^2-2\cdot x\cdot R+R^2-R^2+y^2+z^2=0$$

$$(x-R)^2+y^2+z^2=R^2$$

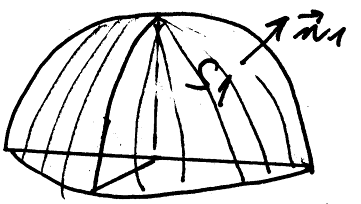
ovo je sfera sa centrom u tački $(R,0,0)$ poluprečnika R

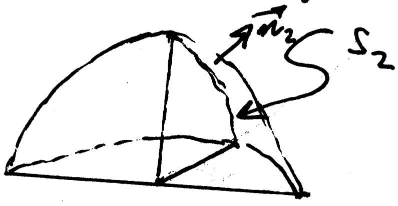


Dati integral podjelimo na tri dijela

$$I = \iint\limits_{S+} y^2\,dy\,dz + \iint\limits_{S+} (y^2+x^2)\,dz\,dx + \iint\limits_{S+} (y^2+x^2+z^2)\,dx\,dy = I_1+I_2+I_3$$

Da bi izračunali I_1 treba nam vektor normale na površ $S+$. Sa slike vidimo da površ $S+$ trebamo podjeliti na dva dijela S_1 i S_2





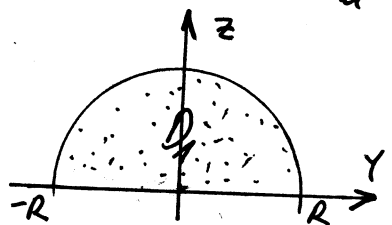
$\measuredangle(\vec{n}_1, x\text{-ose}) \in (0, \pi/2)$

$\Rightarrow \cos\alpha > 0$

$\measuredangle(\vec{n}_2, x\text{-ose}) \in (\pi/2, \pi)$

$\Rightarrow \cos\alpha < 0$

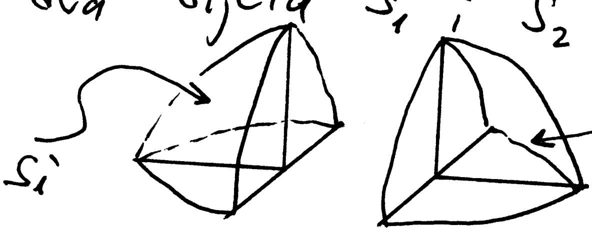
Ortogonalna projekcija sfere na yOz ravan je



$$I_1=\iint\limits_{S+} y^2\,dy\,dz=\iint\limits_{S_1} y^2\,dy\,dz+\iint\limits_{S_2} y^2\,dy\,dz$$

$$\left.\begin{aligned}\iint\limits_{S_1} y^2\,dy\,dz&=+\iint\limits_{D_1} y^2\,dy\,dz\\ \iint\limits_{S_2} y^2\,dy\,dz&=-\iint\limits_{D_1} y^2\,dy\,dz\end{aligned}\right\}\Rightarrow\quad I_1=0$$

Da bi izračunali $I_2=\iint\limits_{S+}(y^2+x^2)\,dz\,dx$, slično kao za I_1, treba nam ugao β između vektora normale $\vec{n}$ i y-ose. Da bi našli ovaj ugao površ S trebamo podjeliti na dva dijela S_1' i S_2'.



za S_1' $\quad \pi/2 \le \beta \le \pi$

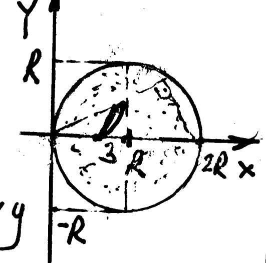
$\cos\beta \le 0$

za S_2' $\quad 0 \le \beta \le \pi/2$

$\cos\beta \ge 0$

$\Rightarrow \quad I_2=0$ (zato što imamo + i − i obe površi imaju istu ortogonalnu projekciju na xOz ravan).

$I_3=\iint\limits_{S+}(x^2+y^2+z^2)\,dx\,dy=$
- ugao između vektora normale $\vec{n}$ na S i z-ose je između 0 i $\pi/2 \Rightarrow \cos\gamma>0$
- ortogonalna projekcija je krug
- $z^2=2Rx-x^2-y^2$



$=$

$(x-R)^2+y^2=R^2$

$x-R=\pm\sqrt{R^2-y^2}$

$=+\iint\limits_{D_3} 2Rx\,dx\,dy=$ uvedimo polarne koordinate
$x=\rho\cos\varphi$
$y=\rho\sin\varphi$
$dx\,dy=\rho\,d\rho\,d\varphi$

$D\xrightarrow{\text{transform.}}D': \begin{cases}0<\rho\le 2R\cos\varphi\\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\end{cases}$ $\quad \cos\varphi=\frac{\rho}{2R}$ $=$

$$=2R\int_{-\pi/2}^{\pi/2}d\varphi\int_0^{2R\cos\varphi}\rho\cos\varphi\,\rho\,d\rho\overset{\text{ZA VJEŽBU}}{=}\ldots=\frac{16R^4}{3}\int_{-\pi/2}^{\pi/2}\cos^4\varphi\,d\varphi\overset{\text{ZA VJEŽBU}}{=}2\pi R^4$$ traženo rješenje